

VARIATIONS OF SELECTIVE SEPARABILITY II: DISCRETE SETS AND THE INFLUENCE OF CONVERGENCE AND MAXIMALITY

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ABSTRACT. A space X is called *selectively separable* (*R-separable*) if for every sequence of dense subspaces $(D_n : n \in \omega)$ one can pick finite (respectively, one-point) subsets $F_n \subset D_n$ such that $\bigcup_{n \in \omega} F_n$ is dense in X . These properties are much stronger than separability, but are equivalent to it in the presence of certain convergence properties. For example, we show that every Hausdorff separable radial space is R-separable and note that neither separable sequential nor separable Whyburn spaces have to be selectively separable. A space is called *d-separable* if it has a dense σ -discrete subspace. We call a space X *D-separable* if for every sequence of dense subspaces $(D_n : n \in \omega)$ one can pick discrete subsets $F_n \subset D_n$ such that $\bigcup_{n \in \omega} F_n$ is dense in X . Although *d-separable* spaces are often also *D-separable* (this is the case, for example, with linearly ordered *d-separable* or stratifiable spaces), we offer three examples of countable non-*D-separable* spaces. It is known that *d-separability* is preserved by arbitrary products, and that for every X , the power $X^{d(X)}$ is *d-separable*. We show that *D-separability* is not preserved even by finite products, and that for every infinite X , the power $X^{2^{d(X)}}$ is not *D-separable*. However, for every X there is a Y such that $X \times Y$ is *D-separable*. Finally, we discuss selective and *D-separability* in the presence of maximality. For example, we show that (assuming $\mathfrak{d} = \mathfrak{c}$) there exists a maximal regular countable selectively separable space, and that (in ZFC) every maximal countable space is *D-separable* (while some of those are not selectively separable). However, no maximal space satisfies the natural game-theoretic strengthening of *D-separability*.

1. INTRODUCTION

The area known as *Selection principles in Mathematics* deals with selective variations of classical topological notions like compactness or separability (see [59] or [53] for a survey and [57] for another survey concentrating on open problems in the field). New results, questions and papers in the area are announced on the periodical SPM bulletin [58]. Looking at the selective version of a certain property adds a combinatorial skeleton to it that often makes it easier to deal with. For example, Leandro Aurichi [10] has recently given one of the few known partial solutions to

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Eric Van Douwen's evasive *D-space problem* (see [28]) by replacing the Lindelöf property with one of its selective strengthenings, the Menger property.

In this paper we will be concerned with the notion of selective separability and its variations. This notion has gained particular attention recently, as witnessed by the papers [52], [44], [16], [17], [18], [11], [12], [50], [34]. A space X is *selectively separable* (also called *M-separable* or *SS*) if for every sequence of dense subspaces $(D_n : n \in \omega)$ one can pick finite $F_n \subset D_n$ so that $\bigcup_{n \in \omega} F_n$ is dense in X . X is *H-separable* if for every sequence of dense subspaces $(D_n : n \in \omega)$ one can pick finite $F_n \subset D_n$ so that every non-empty open set in X intersects all but finitely many F_n . X is *R-separable* if for every sequence of dense subspaces $(D_n : n \in \omega)$ one can pick $p_n \in D_n$ so that $\{p_n : n \in \omega\}$ is dense in X . X is *GN-separable* if X is crowded and for every sequence of dense subspaces $(D_n : n \in \omega)$ one can pick $p_n \in D_n$ so that $\{p_n : n \in \omega\}$ is *groupable*. This means that one can find pairwise disjoint non-empty and finite sets A_m for $m < \omega$ in such a way that $\{p_n : n \in \omega\} = \bigcup \{A_m : m \in \omega\}$ and every non-empty open set in X intersects all but finitely many A_m .

X is SS^+ if Two has a winning strategy in the following game $G_{\text{fin}}(\mathcal{D}, \mathcal{D})$. \mathcal{D} is the collection of all dense subspaces of X . One picks $D_0 \in \mathcal{D}$, then Two picks a finite $F_0 \subset D_0$, then One picks $D_1 \in \mathcal{D}$, etc. Two wins if $\bigcup_{n \in \omega} F_n$ is dense in X . (The term SS^+ is from [12] but the notion and the game $G_{\text{fin}}(\mathcal{D}, \mathcal{D})$ were introduced in [52].) Barman and Dow discovered [12] that every separable Fréchet space is selectively separable. Gruenhage and Sakai [34] pointed out that separable Fréchet spaces are even R-separable, and, if there are no isolated points, GN-separable. In Section 3 we discuss the possibility to extend these results to spaces satisfying convergence-type conditions weaker than Fréchet. It turns out that every regular separable radial space is selectively separable while separable sequential spaces or separable Whyburn spaces need not be selectively separable. We also consider the special case of countably compact spaces.

In Section 4 we consider a weaker form of selective separability: the sets F_n are supposed to be discrete rather than finite; we call this property *D-separability*. This may be also viewed as a natural selective strengthening of the notion of *d-separability*. Recall that X is called *d-separable* [3] (see also [5], [1], [55], [60], [40]) if X has a dense σ -discrete subspace. The notion of *d-separability* is almost as old as separability and was introduced by Kurepa in his Ph.D. dissertation (see also [42]), where it is called *condition K_0* .

It turns out that in some cases d-separable spaces are D-separable. However, the behavior of d-separability and D-separability, is quite different. This is particularly apparent if one looks at the product operation. Every product of d-separable spaces is d-separable [5]; for every T_1 space X , a high enough power of X is d-separable [40] while we show that there are two D-separable spaces with a non-D-separable product, and every (Tychonoff) space has some power which is not D-separable.

In Section 5, we discuss selective separability and D-separability in maximal spaces. For example, we show that (assuming $\mathfrak{d} = \mathfrak{c}$) there exists a maximal countable selectively separable space, and that (in ZFC) every maximal regular countable space is D-separable (while some of those are not selectively separable). However, no maximal space is D^+ -separable (D^+ -separability is a property stronger than D-separability and defined in terms of topological games, see Definition 16 below).

2. TERMINOLOGY AND PRELIMINARIES

For undefined topological notions we refer to [29], while for undefined set-theoretic notions we refer to [37]. The letter X always denotes a topological space. X is *Fréchet* if for every non-closed $A \subset X$ and every $p \in \overline{A} \setminus A$ there is a sequence from A converging to p . X is *sequential* if whenever A is non-closed there are a $p \in \overline{A} \setminus A$ and a sequence from A converging to p . X has *countable tightness* if whenever $p \in \overline{A}$ there is a countable $B \subset A$ such that $p \in \overline{B}$. X has *countable fan tightness* [6] if whenever $p \in \overline{A_n}$ for all $n \in \omega$ one can pick finite $F_n \subset A_n$ so that $p \in \overline{\bigcup_{n \in \omega} F_n}$. X has *countable strong fan tightness* [51] if whenever $p \in \overline{A_n}$ for all $n \in \omega$ one can pick $p_n \in A_n$ so that $p \in \overline{\{p_n : n \in \omega\}}$. X has *dense fan tightness* if X satisfies the definition of fan tightness restricted to A_n dense in X . We will be using the following simple proposition without explicit mention.

Proposition 1. *Let X be separable. Then:*

- (1) [16] *X is selectively separable iff for every $p \in X$ and every sequence $(D_n : n \in \omega)$ of dense subspaces of X one can pick finite sets $F_n \subset D_n$ so that $p \in \overline{\bigcup_{n \in \omega} F_n}$ (in other words, X has countable tightness with respect to dense sets).*
- (2) *X is H -separable iff for every $p \in X$ and every sequence $(D_n : n \in \omega)$ of dense subspaces of X one can pick finite sets $F_n \subset D_n$ so that every neighborhood of p meets all but finitely many F_n .*
- (3) *X is R -separable iff for every $p \in X$ and every sequence $(D_n : n \in \omega)$ of dense subspaces of X one can pick points $p_n \in D_n$ so that $p \in \overline{\{p_n : n \in \omega\}}$ (in other words, X has countable strong tightness with respect to dense sets).*
- (4) *X is GN -separable iff for every $p \in X$ and every sequence $(D_n : n \in \omega)$ of dense subspaces of X one can pick points $p_n \in D_n$ and represent $\{p_n : n \in \omega\} = \bigcup_{m \in \omega} A_m$ where the sets A_m are non-empty, finite, and pairwise disjoint, so that every neighborhood of p intersects all but finitely many A_m .*

$\delta(X) = \sup\{d(D) : D \text{ is dense in } X\}$. If X is compact, then $\delta(X) = \pi w(X)$ [38]. Obviously, $\delta(X) = \omega$ for every selectively separable space X .

X is *radial* if for every $A \subset X$ and every $p \in \overline{A}$ there is a well-ordered net $\{x_\alpha : \alpha < \kappa\} \subset A$ which converges to p . X is *pseudoradial* if for every non-closed $A \subset X$ there is a $p \in \overline{A} \setminus A$ and a well-ordered net $\{x_\alpha : \alpha < \kappa\} \subset A$ which converges to p . A set $A \subset X$ is κ -closed (where κ is a cardinal) if $\overline{B} \subset A$ whenever $B \subset A$ and $|B| \leq \kappa$. X is *semiradial* (see [15], [22]) if for every κ , every non- κ -closed set A contains a well-ordered net of length $\leq \kappa$ converging to a point outside A . Among the various subclasses of pseudoradial spaces considered in the literature, the class of semiradial spaces is the smallest one which includes all radial and all sequential spaces.

X has the *Whyburn property* if for every $A \subset X$ and every $p \in \overline{A} \setminus A$ there exists $B \subset A$ such that $\overline{B} = A \cup \{p\}$. Every Fréchet space is Whyburn and every compact Whyburn space is Fréchet (see [61]). For $p \in \omega^*$, the space $\omega \cup \{p\}$ with the topology inherited from $\beta\omega$ is a non-Fréchet Whyburn topological space. The space $C_p([0, 1])$ of all continuous functions from $[0, 1]$ to \mathbb{R} with the topology of pointwise convergence is a nice Whyburn topological group which is not Fréchet [23].

Recall that for $M \subset X$, $\text{seqcl}(M) = \{x \in X : \text{there is a sequence converging from } M \text{ to } x\}$ and $\text{seqcl}_\alpha(M)$ is defined inductively by $\text{seqcl}_\alpha(M) = \text{seqcl}(\bigcup_{\beta < \alpha} \text{seqcl}_\beta(M))$. If X is sequential then there exists an ordinal α^* called the *sequential order* of X

such that $\text{seqcl}_{\alpha^*}(M) = \overline{M}$ for every $M \subset X$. The sequential order of any sequential space is $\leq \omega_1$.

Let ${}^n\omega$ be the set of all functions $s : n = \{0, 1, \dots, n-1\} \rightarrow \omega$ and let $\text{Seq} = \bigcup \{{}^n\omega : n < \omega\}$. If $s \in {}^n\omega$ and $k \in \omega$, we write $s \smallfrown k = s \cup \{(n, k)\} \in {}^{n+1}\omega$. Given a free filter \mathcal{F} on ω , we denote by $\text{Seq}(\mathcal{F})$ the topological space having Seq as the underlying set and the topology obtained by declaring a set $U \subseteq \text{Seq}$ open if and only if for any $s \in U$ $\{n : s \smallfrown n \in U\} \in \mathcal{F}$. $\text{Seq}(\mathcal{F})$ is always a Hausdorff zero-dimensional dense-in-itself space (see [63] for more information). In several instances in this paper, we will use $\text{Seq}(\mathcal{F})$ where $\mathcal{F} = \{A \subset \omega : |\omega \setminus A| < \omega\}$ is the Fréchet filter. This space is also known under the name S_ω [9]. $\text{Seq}(\mathcal{F})$ is sequential of sequential order ω_1 .

X is *crowded* (also called *dense in itself*) if X does not have isolated points. X is *maximal* if X is crowded and no topology strictly stronger than the topology of X is crowded. X is *resolvable* (ω -*resolvable*) if X contains two (respectively, a countably infinite family of pairwise) disjoint dense subspaces. X is *submaximal* if every subset is open in its closure, or, equivalently (see [8]) if the complement of every dense set is closed and discrete. Every maximal space is submaximal. Every crowded submaximal (hence every maximal) space is irresolvable (= not resolvable). X is *Baire* if no non-empty open set in X is representable as the union of countably many nowhere dense sets. X is *strongly irresolvable* [41] if all non-empty open sets are irresolvable. Strongly irresolvable Baire is abbreviated as SIB [41]. For any space X , the *dispersion character* $\Delta(X)$ of X is defined as the minimum cardinality of a non-empty open set in X . A crowded space X is *extra-resolvable* if there is a family \mathcal{G} of dense subspaces of X such that $|\mathcal{G}| > \Delta(X)$ and for every two distinct $G, G' \in \mathcal{G}$, $G \cap G'$ is nowhere dense.

X is *discretely generated* [27] if whenever $p \in \overline{A}$ there is a discrete $D \subset A$ such that $p \in \overline{D}$.

X is a σ -*space* if X has a σ -discrete network.

X is *monotonically normal* if one can assign to every point $x \in X$ and open set $U \subset X$ an open set $H(x, U) \subset U$ such that $x \in H(x, U)$ and if $H(x, U) \cap H(y, V) \neq \emptyset$ then either $x \in V$ or $y \in U$. The function H is called a *monotone normality operator*.

X is *stratifiable* if one can assign to every $n \in \omega$ and every closed set $H \subset X$ an open set $G(n, H) \supset H$ so that $H = \bigcap_{n \in \omega} \overline{G(n, H)}$ and $G(n, H) \subset G(n, K)$ whenever $H \subset K$. Every stratifiable space is both monotonically normal and a σ -space [32].

A set $D \subset X^2$ is called *slim* [33] if the intersection of D with every *cross-section* $(\{p\} \times X) \cup (X \times \{p\})$ is nowhere dense (in this cross section). If $B \subset A$ then $\pi_B : \prod_{\alpha \in A} X_\alpha \rightarrow \prod_{\alpha \in B} X_\alpha$ denotes the projection of the product onto a subproduct.

$\text{cov}(\mathcal{M})$ is the minimum cardinality of a family of nowhere dense subsets of \mathbb{R} that covers \mathbb{R} . A function $g \in \omega^\omega$ is said to *guess* the family of functions $\Phi \subset \omega^\omega$ if for every $f \in \Phi$, $f(n) = g(n)$ for infinitely many n . It is known that if $|\Phi| < \text{cov}(\mathcal{M})$, then there is g that guesses Φ , see [14].

3. CONVERGENCE AND SELECTIVE SEPARABILITY

Gruenhage and Sakai [34] observed that separable Fréchet spaces are R-separable. Basically, there are three “natural ways” to try to strengthen this result: one is

to move from Fréchet to radial, another is to move from Fréchet to sequential (or, more generally, to spaces of countable tightness), and yet another is from Fréchet to Whyburn.

3.1. Radial spaces.

Proposition 2. (1) *A Hausdorff separable radial (with respect to dense subspaces) space X is R -separable.*

(2) *If, in addition, X does not have isolated points, then X is GN -separable.*

Proof. (1) It suffices to show that, given a point $p \in X$ and a sequence of dense subspaces $(D_n : n \in \omega)$ one can pick $p_n \in D_n$ so that $p \in \overline{\{p_n : n \in \omega\}}$. Assume p is not isolated, otherwise p is contained in every dense set and the statement we want to prove becomes trivial. Let \mathcal{U} be a maximal pairwise disjoint family of non-empty open sets in X such that $x \notin \overline{U}$ for every $U \in \mathcal{U}$ (*). Since X is Hausdorff and p is not isolated, $\bigcup \mathcal{U}$ is dense in X . Since X is separable, \mathcal{U} is countable; enumerate it as $(U_n : n \in \omega)$. Put $Y = \bigcup \{D_n \cap U_n : n \in \omega\}$. Then Y is dense in X and thus there is $S \subset Y$ which can be enumerated as a well-ordered net converging to p . We can assume that S is of minimal cardinality among all well-ordered nets contained in Y which converge to p , so $|S|$ is regular, and then it follows from (*) and the countability of \mathcal{U} that S is a convergent sequence. Again from (*), S must have non-empty intersection with infinitely many sets $D_n \cap U_n$; pick a subsequence $S' \subset S$ that intersects each $D_n \cap U_n$ at at most one point.

For $n \in \omega$, if $S' \cap D_n \cap U_n$ is non-empty, let p_n be the unique point in this intersection. Otherwise pick p_n arbitrarily. Then the points p_n are as desired.

(2) Partition $\{p_n : n \in \omega\}$ from part 1 into pairwise disjoint finite sets A_m so that each A_m contains at least one point of S' and apply Proposition 1, part 4. \square

The above proposition cannot be extended to pseudoradial spaces because as we will see below even separable sequential spaces need not be selectively separable.

Corollary 3. *Every compact separable radial space has countable π -weight.*

Proof. For a selectively separable X , $\delta(X) = \omega$, and for a compact X , $\delta(X) = \pi w(X)$ [38]. \square

Corollary 4. *Every separable compact monotonically normal space has countable π -weight.*

Proof. Monotonically normal spaces are radial [64]. \square

While we could not find a reference for Corollary 3, Corollary 4 can be also derived from [31], Corollary 19 (which says that density equals π -weight for monotonically normal compact spaces).

One can wonder whether Corollary 3 can be extended to semiradial spaces. Some mild evidence is provided by the fact that every compact sequential separable space has countable π -weight (this follows easily from the inequality $\pi\chi(X) \leq t(X)$, for every compact space X see [36]) and every sequential space is semiradial. However, the answer is consistently negative. Bella [15] showed that the space 2^{ω_1} is semiradial if and only if $\mathfrak{p} > \omega_1$. One might hope at least for a consistency result. Indeed, A. Dow proved [26] that there are models in which every compact separable radial space is Fréchet, but it is still an open problem if there are models in which

every compact separable semiradial space is sequential. In view of Corollary 3, we suggest a weaker form of this problem.

Question 5. Is it consistent that every compact separable semiradial space has countable π -weight?

3.2. Sequential spaces and spaces of countable tightness. Strengthening the Barman-Dow result in this direction is in general not possible.

If K is an infinite subset of ω , then it is easy to check that the set $D_K = \bigcup\{^k\omega : k \in K\}$ is dense in $\text{Seq}(\mathcal{F})$. Moreover, it is also quite easy to realize that, for any choice of a finite set $F_n \subseteq {}^n\omega$, the set $\bigcup\{F_n : n < \omega\}$ is closed and nowhere dense in $\text{Seq}(\mathcal{F})$. Taking this into account, we see that if $\{H_n = \bigcup\{^k\omega : n \leq k < \omega\}\}$, then the sequence of dense sets $\{H_n : n < \omega\}$ witnesses that the space $\text{Seq}(\mathcal{F})$ is never selectively separable.

If as \mathcal{F} we take the filter of all cofinite subsets of ω , then $\text{Seq}(\mathcal{F})$ turns out to be sequential. Indeed, if A is a non-closed subset of $\text{Seq}(\mathcal{F})$, then $\text{Seq}(\mathcal{F}) \setminus A$ is not open and so there is some $s \in \text{Seq}(\mathcal{F}) \setminus A$ such that the set $\{n : s \cap n \in \text{Seq}(\mathcal{F}) \setminus A\}$ has an infinite complement E . Therefore, $S = \{s \cap n : n \in E\} \subseteq A$ and we immediately check that S converges to s .

Thus, there is a countable sequential space which is not selectively separable. However, this space has sequential order ω_1 . So, what is left open is:

Question 6. Is every Hausdorff separable sequential space of finite or countable sequential order selectively separable?

That separable spaces of countable tightness need not be selectively separable is well known. There are many examples such as $C_p(\text{Irrationals})$ or even some countable spaces [16], [17]. However, adding some restrictions on the character of points or some covering properties we can get positive results. Here are a few of the most interesting.

Proposition 7.

- (1) [16] *If a separable space X of countable tightness has a dense set of points of character less than \mathfrak{d} , then X is selectively separable.*
- (2) *A regular countably compact separable space of countable tightness is selectively separable.*
- (3) [34] *More generally, let X be a regular separable space of countable tightness. If each point is contained in a countably compact set of countable character in X , then X is selectively separable.*

Item 2 follows directly from the fact that a regular countably compact space of countable tightness has countable fan tightness [7]. In order to slightly improve this result we prove a proposition which may be of some independent interest.

Proposition 8. *A regular countably compact space X of countable tightness has countable strong fan tightness.*

Proof. Let $b \in X$, $A_n \subset X$, $b \in \overline{A_n}$ for all $n \in \omega$. Without loss of generality we assume that the sets A_n are countable. Put $\tilde{X} = \overline{\bigcup_{n \in \omega} A_n}$. Then \tilde{X} is a regular separable countably compact space of countable tightness; $b \in \tilde{X}$. Fix a base \mathcal{B} of neighborhoods of b in \tilde{X} such that $|\mathcal{B}| \leq \mathfrak{c}$. For every $U \in \mathcal{B}$ fix an open in \tilde{X} set $V(U)$ such that $b \in V(U) \subset \overline{V(U)} \subset U$. Last, fix an almost disjoint family \mathcal{R} of infinite subsets of ω enumerated by \mathcal{B} : $\mathcal{R} = \{N_U : U \in \mathcal{B}\}$.

Let $U \in \mathcal{B}$. For every $n \in N_U$ pick $x_{n,U} \in A_n \cap V(U)$. Since \tilde{X} is countably compact the set $\{x_{n,U} : n \in N_U\}$ has a limit point, say s_U . Then $s_U \in \overline{V(U)} \subset U$.

Put $S = \{s_U : U \in \mathcal{B}\}$. Then $b \in \overline{S}$. Since \tilde{X} has countable tightness there is a countable subfamily $\mathcal{B}_0 \subset \mathcal{B}$ such that $b \in \overline{S_0}$ where $S_0 = \{s_U : U \in \mathcal{B}_0\}$. Enumerate $\mathcal{B}_0 = \{U_m : m \in \omega\}$. For $U_m \in \mathcal{B}_0$ put $\tilde{N}_{U_m} = N_{U_m} \setminus \bigcup_{k < m} N_{U_k}$. Then the sets \tilde{N}_{U_m} are pairwise disjoint, and \tilde{N}_{U_m} differs from N_{U_m} only in finitely many points.

Let $n \in \omega$. If $n \in \tilde{N}_{U_m}$ for some (single!) m then put $a_n = x_{n,U_m}$. Otherwise pick $a_n \in A_n$ arbitrarily. Thus we have $a_n \in A_n$ defined for all n , and $b \in \overline{\{a_n : n \in \omega\}}$ (because $b \in \overline{S_0}$ and each point of S_0 is in $\overline{\{a_n : n \in \omega\}}$). \square

As an immediate corollary we get that regular countably compact separable spaces of countable tightness are R-separable. However, we are going to prove a stronger result which is a simultaneous improvement of all parts of Proposition 7. Let X be a space and $x \in X$. Weakening the definition of the cardinal function $h(x, X)$ [29] we denote by $h^*(x, X)$ the smallest cardinal number κ such that there exists a countably compact $H \subset X$ with $x \in H$ and $\chi(H, X) = \kappa$.

Theorem 9. *Let X be a regular separable space of countable tightness.*

- (a) *If the inequality $h^*(x, X) < \mathfrak{d}$ holds in a dense set of points, then X is selectively separable.*
- (b) *If the inequality $h^*(x, X) < \text{cov}(\mathcal{M})$ holds in a dense set of points, then X is R-separable.*

Proof. Without any loss of generality, we may suppose that X does not have isolated points. Let $(D_n : n \in \omega)$ be a sequence of dense subsets of X . Since from the hypotheses it follows that any dense set is separable, we may assume that each D_n is countable.

Part a. Let us begin by fixing a countable dense set C such that $h^*(x, X) < \mathfrak{d}$ holds for each $x \in C$. Let $\{J_x : x \in C\}$ be a partition of ω in infinite sets and fix for each $x \in C$ a countably compact subspace H_x satisfying $x \in H_x$ and $\chi(H_x, X) = \kappa_x < \mathfrak{d}$. Let $\Phi = \prod_{n \in J_x} [D_n]^{<\omega}$ and for each $\phi \in \Phi$ let $[\phi] = D(\bigcup\{\phi(n) : n \in J_x\})$ (here $D(S)$ indicates the derived set of S).

Claim. $x \in \overline{\bigcup\{[\phi] : \phi \in \Phi\}}$.

Proof of Claim. Fix an open neighborhood V of x . We are going to check that $\overline{V} \cap \bigcup\{[\phi] : \phi \in \Phi\} \neq \emptyset$. Case 1 If there exists an infinite set $J \subseteq J_x$ such that $V \cap D_n \cap H_x$ is infinite for each $n \in J$, then we may define a function $\psi \in \Phi$ by letting $\psi(n) = \{p_n\}$, for some $p_n \in V \cap D_n \cap H_x$ if $n \in J$, and $\psi(n) = \emptyset$ otherwise. The function ψ can be made one-to-one on J and so $\bigcup\{\psi(n) : n \in J_x\}$ is an infinite subset of the countably compact set H_x . This guarantees that $[\psi] \neq \emptyset$ and clearly by construction $[\psi] \subseteq \overline{V}$. Case 2 If such a J does not exist, then we may assume, without any loss of generality, that $V \cap D_n \cap H_x = \emptyset$ for each $n \in J_x$ (it suffices to replace D_n with $D_n \setminus (V \cap H_x)$ for all but finitely many $n \in J_x$ and then remove a finite part of J_x). For any $n \in J_x$ write $D_n = \{x_{n,k} : k < \omega\}$ and let $\{U_\alpha : \alpha \in \kappa_x\}$ be a local base of H_x in X . For any α fix a function $f_\alpha : J_x \rightarrow \omega$, defined in such a way that $x_{n,f_\alpha(n)} \in V \cap U_\alpha$ for each $n \in J_x$. Since $\kappa_x < \mathfrak{d}$, there is a function $g : J_x \rightarrow \omega$ such that for each α the inequality $f_\alpha(n) \leq g(n)$ holds for infinitely many n 's. Let ψ be the element of Φ defined by letting $\psi(n) = \{x_{n,k} : k \leq g(n)\}$.

By the choice of g , for each α there is some $n \in J_x$ such that $U_\alpha \cap \psi(n) \neq \emptyset$ and this in turn implies that we must have $\overline{\bigcup\{\psi(n) : n \in J_x\}} \cap H_x \neq \emptyset$. Since the values of ψ are disjoint from H_x , the latter formula implies that $[\psi] \neq \emptyset$. Of course, by construction we have $[\psi] \subseteq \overline{V}$ and the claim is proved. \triangle

Now, thanks to the Claim and the countable tightness of X , there are countably many $\phi_n \in \Phi$ and points $z_n \in [\phi_n]$ such that $x \in \overline{\{z_n : n \in \omega\}}$. Observe that if O is an open neighborhood of x then $z_n \in O$ for some n and, being z_n an accumulation point of the set $\bigcup\{\phi_n(k) : k \in J_x\}$, we actually have $O \cap \bigcup\{\phi_n(k) : k \in J_x, k \geq n\} \neq \emptyset$. Now, we put $F_n^x = \bigcup\{\phi_n(k) : k \leq n\}$. As in the first paragraph, we have $x \in \overline{\bigcup\{F_n^x : n \in J_x\}}$. To finish, it is enough to note that the chosen family $\{F_n^x : n \in J_x, x \in C\}$ is dense in X .

Part b. Fix a countable dense set C such that $h^*(x, X) < cov(\mathcal{M})$ holds for each $x \in C$. Let $\{J_x : x \in C\}$ be a partition of ω in infinite sets and fix for each $x \in C$ a countably compact subspace H_x satisfying $x \in H_x$ and $\chi(H_x, X) = \kappa_x < cov(\mathcal{M})$.

Let $x \in C$. Fix a base \mathcal{B} of neighborhoods of x in X such that $|\mathcal{B}| \leq \mathfrak{c}$. Assign to every $U \in \mathcal{B}$ an open neighborhood V_U of x so that $x \in V_U \subset \overline{V_U} \subset U$. Fix an almost disjoint family $\{N_{x,U} : U \in \mathcal{B}\}$ of infinite subsets of ω .

Claim. For every $U \in \mathcal{B}$ there is $\phi \in \prod_{n \in N_{x,U}} D_n$ such that $\overline{V_U} \cap D(\{\phi(n) : n \in N_{x,U}\}) \neq \emptyset$.

Proof of Claim. In the case when there exists an infinite $J \subset N_{x,U}$ such that $V_U \cap D_n \cap H_x$ is infinite for each $n \in J$ the argument repeats the similar one from part a. So assume no such J exists and then without loss of generality we assume that $V_U \cap D_n \cap H_x = \emptyset$ for all $n \in N_{x,U}$. Enumerate $D_n = \{x_{n,k} : k \in \omega\}$. Let $\{O_\alpha : \alpha < \kappa_x\}$ (where $\kappa_x < cov(\mathcal{M})$) be a local base of H_x in X . For every α fix a function $f_\alpha : N_{x,U} \rightarrow \omega$ defined in such a way that $x_{n,f_\alpha(n)} \in V_U \cap O_\alpha$ for each $n \in N_{x,U}$. Since $\kappa_x < cov(\mathcal{M})$ there is a function $g : N_{x,U} \rightarrow \omega$ such that for each α the equation $f_\alpha(n) = g(n)$ holds for infinitely many n 's. Define ϕ by $\phi(n) = x_{n,g(n)}$ for each $n \in N_{x,U}$. This proves the claim. \triangle

Now, using claim, for every $U \in \mathcal{B}$ fix $\phi_U \in \prod_{n \in N_{x,U}} D_n$ and $z_U \in \overline{V_U} \cap D(\{\phi_U(n) : n \in N_{x,U}\})$. Put $Z = \{z_U : U \in \mathcal{B}\}$. Then $x \in \overline{Z}$ and, since X has countable tightness, there is a countable subset, say $Z_0 = \{z_{U_k} : k \in \omega\}$, such that $x \in \overline{Z_0}$. For $k \in \omega$, put $\tilde{N}_{x,U_k} = N_{x,U_k} \setminus \bigcup_{l < k} N_{x,U_l}$. Then the sets \tilde{N}_{x,U_k} ($k \in \omega$) are pairwise disjoint and differ from N_{x,U_k} only by finitely many elements. Let $n \in J_x$. If n belongs to some (then only to one) N_{x,U_k} then put $a_n = x_{n,g(n)}$. If not, choose $a_n \in D_n$ arbitrarily. Then $x \in \overline{\{a_n : n \in J_x\}}$. \square

A crucial role in the proof of Theorem 9 as well as in Gruenhage and Sakai's proof of Proposition 7, part 3 [34] is played by Proposition 7, part 2 (and its variation, Proposition 8). This suggests to look for some possible generalization. In one direction we may try to weaken "regular" to Hausdorff and in the other to weaken countably compact to pseudocompact. Unfortunately, we have an answer only for the first case.

It is well known (see [63]) that if \mathcal{U} is a free ultrafilter on ω then the space $Seq(\mathcal{U})$ is extremally disconnected. So, $X = Seq(\mathcal{U})$ is a countable Hausdorff zero-dimensional extremally disconnected non-selectively separable space. Now, consider

the Čech-Stone compactification βX of this X . Theorem 1.1g of [62] shows that there exists a strengthening of the topology of βX in such a way that the resulting space Y has the following properties:

- (1) X is a dense subspace of Y ;
- (2) Y is locally countable;
- (3) each closed infinite subset of Y has cardinality 2^c .

So, we get:

Example 10. There exists a separable countably compact Urysohn space of countable tightness which is not selectively separable.

Moving from countably compact to pseudocompact appears much harder. Indeed, with a lot of effort, Bella and Pavlov [20] constructed a Tychonoff pseudocompact space of countable tightness which does not have countable fan tightness. But such a space has a countable set of isolated points and so it is selectively separable. For these reasons, the next problem sounds very interesting:

Problem 11. Find a Tychonoff pseudocompact separable space of countable tightness which is not selectively separable.

Forgetting separability, we may formulate a possibly easier problem:

Problem 12. Find a Tychonoff pseudocompact space of countable tightness which does not have countable dense fan tightness.

3.3. Whyburn spaces. Barman and Dow constructed a countable regular maximal space which is not selectively separable [12]. On the other hand, the first author and I. Yaschenko showed in [23] that every regular maximal space has the Whyburn property. Therefore we get:

Corollary 13. *There exists a countable regular Whyburn space which is not selectively separable.*

Tkachuk and Yaschenko [61] proved that every countably compact Whyburn space is Fréchet. So countably compact Whyburn separable spaces are selectively separable. However pseudocompact Whyburn spaces need not be Fréchet [48], not even if they have countable tightness [21]. So, also in view of Problem 12, we have the following question.

Question 14. *Suppose X is a pseudocompact Whyburn separable space. Is X selectively separable? What if X has countable tightness?*

4. D-SEPARABILITY

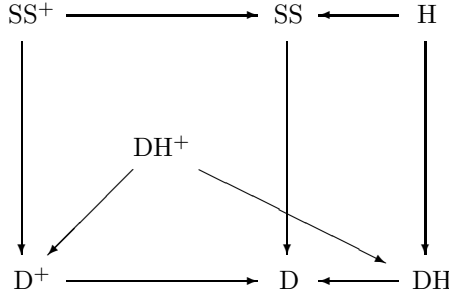
A space is called d -separable if it contains a σ -discrete dense subspace. We introduce some selective version of this property.

Definition 15. X is D -separable if for every sequence of dense subspaces $(D_n : n \in \omega)$ one can pick discrete sets $F_n \subset D_n$ so that $\bigcup_{n \in \omega} F_n$ is dense in X . X is DH -separable if for every sequence of dense subspaces $(D_n : n \in \omega)$ one can pick discrete $F_n \subset D_n$ so that every non-empty open set in X intersects all but finitely many F_n .

Consider the following games on a space X (as above, \mathcal{D} denotes the collection of all dense subspaces of X). In the game $G_{\text{dis}}(\mathcal{D}, \mathcal{D})$, One picks $D_0 \in \mathcal{D}$, then Two picks a discrete $F_0 \subset D_0$, then One picks $D_1 \in \mathcal{D}$, etc. Two wins if $\bigcup_{n \in \omega} F_n$ is dense in X . The game $G_{\text{dis}, H}(\mathcal{D}, \mathcal{D})$ is similar, only Two wins if every non-empty open set in X intersects all but finitely many F_n .

Definition 16. X is D^+ -separable if Two has a winning strategy in $G_{\text{dis}}(\mathcal{D}, \mathcal{D})$. Say that X is DH^+ -separable if Two has a winning strategy in $G_{\text{dis}, H}(\mathcal{D}, \mathcal{D})$.

The following implications (where, for example, SS denotes selectively separable and D denotes D-separable) are straightforward.



We know that most of the arrows in the diagram cannot be reversed. To see that an arrow pointing from a selective separability-type property (top row of the diagram) to a selective d -separability-type property (bottom row and center of the diagram) cannot be reversed simply take any metric non-separable space. In some cases we will be able to improve this and obtain a separable counterexample. For instance, Example 25 is a countable space showing that $D \not\leftrightarrow SS$. Any countable submaximal space shows that $D \not\leftrightarrow D^+$ (see Corollary 69 and Theorem 76). However, the relationship between D and DH is not well-understood, and hence we have the following open problem

Problem 17.

- (1) Find an example of a (countable) D -separable non- DH -separable space.
- (2) Find an example of a (countable) D^+ -separable non- DH^+ -separable space.
- (3) Find an example of a (countable) DH -separable, non- DH^+ -separable space.

4.1. Which spaces are D-separable?

Proposition 18. (1) Every space with a σ -disjoint π -base is DH^+ -separable.
 (2) Every space with a σ -locally finite π -base is DH^+ -separable.
 (3) Every T_1 space with a σ -closure preserving π -base is DH^+ -separable.

Lemma 19. If \mathcal{U} and \mathcal{V} are pairwise disjoint families of non-empty sets in X , then there is a pairwise disjoint family \mathcal{W} of non-empty sets in X such that:

- (1) Every element of \mathcal{U} contains an element of \mathcal{W} ;
- (2) Every element of \mathcal{V} contains an element of \mathcal{W} ;
- (3) Every element of \mathcal{W} is contained in some element of $\mathcal{U} \cup \mathcal{V}$.

Proof. Put $\mathcal{W}_0 = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V} \text{ and } U \cap V \neq \emptyset\}$ and $\mathcal{W} = \mathcal{W}_0 \cup \{U \in \mathcal{U} : \text{there is no } W \in \mathcal{W} \text{ with } U \supset W\} \cup \{V \in \mathcal{V} : \text{there is no } W \in \mathcal{W} \text{ with } V \supset W\}$. \square

Lemma 20. *If \mathcal{U} is a locally finite family of non-empty open sets in a space X and D is a dense subspace of X , then there is a discrete set A such that $A \cap U \neq \emptyset$ for every non-empty $U \in \mathcal{U}$.*

Proof. ¹ For every $U \in \mathcal{U}$ pick a point $p_U \in U \cap D$. Let $B = \{p_U : U \in \mathcal{U}\}$. The local finiteness of \mathcal{U} implies that every $x \in B$ is contained in a set $V_x \subset B$ which is finite and open in B . We choose as V_x an open set of minimum size. Let us call a point $x \in B$ good if for each $y \in V_x$ we have $V_y = V_x$. It is clear that for each $x \in B$ there is a good point y such that $V_y \subset V_x$. Moreover, if y and z are good, then either $V_y = V_z$ or $V_y \cap V_z = \emptyset$. Now, fix a well ordering on B and for each good point $y \in B$ let $a(y) = \min V_y$. The set A of all such $a(y)$ is discrete and A intersects every element of \mathcal{U} . \square

Proof of Proposition 18. (1) Let $\mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n$ (where each \mathcal{U}_n is pairwise disjoint and consists of non-empty sets) be a π -base of X . Applying Lemma 19 inductively one gets pairwise disjoint families \mathcal{W}_n of non-empty open sets such that whenever $m \leq n$, every element of \mathcal{W}_n is contained in some element of \mathcal{U}_m , and every element of \mathcal{U}_m contains an element of \mathcal{W}_n . At the n th inning One chooses a dense subspace D_n and Two picks for every $U \in \mathcal{U}_n$ a point $p_{D_n, U} \in U$ and sets $F_n = \{p_{D_n, U} : U \in \mathcal{U}_n\}$.

(2) Let $\mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n$ (where each \mathcal{U}_n is locally finite) be a π -base of X . Put $\mathcal{W}_n = \bigcup_{m \leq n} \mathcal{U}_m$. At the n th inning One chooses a dense subspace D_n and Two uses Lemma 20 to find a discrete subspace $F_n \subset D_n$ which meets every element of \mathcal{W}_n .

(3) Let $\mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n$ (where each \mathcal{U}_n is closure preserving) be a π -base of X . At the n th inning One chooses a dense subspace D_n and Two picks for every $U \in \mathcal{U}_n$ a point $p_{D_n, U} \in U$ and sets $F_n = \{p_{D_n, U} : U \in \mathcal{U}_n\}$. The family $\{\{p_{D_n, U} : U \in \mathcal{U}_n\} : n \in \omega\}$ is closure preserving. So, since X is T_1 , the set $F_n = \{p_{D_n, U} : U \in \mathcal{U}_n\}$ is discrete. \square

In particular, every metrizable space (or, more generally, every T_1 M1-space (= a space with a σ -closure preserving base) is DH^+ -separable. But below we will see more: every M3 (= stratifiable) space is DH^+ -separable.

Shapirovsii showed [55] that every space with a σ -point finite base is d-separable.

Question 21. Is every space with a σ -point finite base D-separable? If yes, how about other properties in Definitions 15 and 16?

Proposition 22. *Let X be a collectionwise Hausdorff discretely generated space with a σ -closed discrete dense set. Then X is DH^+ -separable.*

Proof. Let $H = \bigcup_{n \in \omega} H_n$ be dense in X (where each H_n is closed and discrete). Without loss of generality we assume that $H_n \subset H_m$ whenever $n \leq m$. For every n , fix a pairwise disjoint open expansion $\{U_{n,x} : x \in H_n\}$ of H_n .

At the n th inning ONE picks a dense $D_n \subset X$. Then Two, for every $x \in H_n$, picks a discrete $F_{n,x} \subset D_n \cap U_{n,x}$ such that $x \in \overline{F_{n,x}}$ and sets $F_n = \bigcup_{x \in H_n} F_{n,x}$. \square

Corollary 23. *Every monotonically normal σ -space is DH^+ -separable.*

¹In the T_1 case, the proof is trivial: just pick a point in every non-empty element of \mathcal{U} . But the statement is valid without any assumption on separation.

Proof. Every σ -space has a σ -closed discrete dense set, and every monotonically normal space is both collectionwise Hausdorff (see [32]) and discretely generated [27]. \square

Corollary 24. *Every stratifiable space is DH^+ -separable.*

Proof. Because a stratifiable space is both monotonically normal and a σ -space. \square

Boaz Tsaban asked us in private communication whether a separable D-separable space has to be selectively separable. This can be disproved by taking the space $Seq(\mathcal{F})$ where \mathcal{F} is any ultrafilter on ω . Indeed, this space is countable, and hence it is trivially a σ -space. Moreover, it is monotonically normal by Theorem 3.2 of [39]. If \mathcal{F} is a Ramsey ultrafilter (which exists, for example, if one assumes CH), then $Seq(\mathcal{F})$ is even a topological group (see [63]). So we arrive to the following theorem:

Theorem 25. *There is a countable DH^+ -separable space X which is not selectively separable. Under CH the space X can even be taken to be a topological group.*

Yet the following is still unknown.

Question 26. Is there a compact separable D -separable non-selectively separable space?

Monotone normality alone does not imply D -separability. Indeed, it suffices to consider a Suslin Line \mathbb{L} . \mathbb{L} is monotonically normal because it is linearly ordered, and it cannot even have a σ -discrete dense set because every discrete set in \mathbb{L} is countable, but \mathbb{L} is not separable. Moreover, it is easy to see that for linearly ordered spaces, the three properties: d -separability, D -separability and having a σ -discrete π -base, are equivalent. This motivates the following question:

Question 27. Is it true that a monotonically normal space is D -separable if and only if it is d -separable?

We conclude the section with a partial positive result. The principal tool in the proof of it is a theorem by Gartside stating that $\pi w(X) = d(X) = hd(X)$ for X having a monotonically normal compactification [31]

Theorem 28. *Suppose a space X has a monotonically normal compactification. Then the following conditions are equivalent:*

- (1) X is d -separable;
- (2) X is D -separable;
- (3) X has a σ -disjoint π -base.

Proof. Of course it is enough to prove (1) \Rightarrow (3). Call a non-empty open set *poor* if for every non-empty open $V \subset U$, $\pi w(V) = \pi w(U)$. It is clear that every non-empty open set contains a poor set and thus in every topological space X one can find a pairwise disjoint family of poor sets \mathcal{U} such that $X = \bigcup \mathcal{U}$. Note that X is d -separable, or is D -separable, or is a space with a σ -disjoint π -base iff so is every element of \mathcal{U} . Moreover, if X has a monotonically normal compactification, then also every element of \mathcal{U} does. So, without loss of generality we can assume that X itself is poor.

So let X be a poor space with a monotonically normal compactification and monotone normality operator H , and let $D = \bigcup_{n \in \omega} D_n$ be dense in X where each

D_n is discrete. Without loss of generality, we can assume that the sets D_n are pairwise disjoint. Let \mathcal{P} be a π -base of X of cardinality $\pi w(X) = \kappa$. By Gartside's theorem (applied to the subspace U) and poverty of X , $|U \cap D| = \kappa$ for every $U \in \mathcal{P}$. Then, enumerating \mathcal{P} and D , one easily defines an injection $f : \mathcal{P} \rightarrow D$ such that $f(U) \in U$ for every $U \in \mathcal{P}$. For $n \in \omega$, put $\mathcal{P}_n = \{U \in \mathcal{P} : f(U) \in D_n\}$ and let $\{V(f(U)) : U \in \mathcal{P}_n\}$ be a family of open sets such that $V(f(U)) \cap D_n = \{f(U)\}$ and $V(f(U)) \subset U$ for every $U \in \mathcal{P}_n$. For $U \in \mathcal{P}_n$, put $U' = H(f(U), V(f(U)))$. By the properties of the monotone normality operator H , the family $\mathcal{P}'_n = \{U' : U \in \mathcal{P}_n\}$ is pairwise disjoint and hence $\bigcup_{n < \omega} \mathcal{P}'_n$ is a σ -disjoint π -base for X . \square

4.2. Subspaces, unions. The following is straightforward:

Proposition 29. (1) Every open subspace, as well as every dense subspace, of a space with one of the properties from Definitions 15 and 16 has the same property.

(2) If X has an open dense subspace with one of the properties from Definitions 15 and 16 then X has the same property.

(3) A discrete sum of spaces with one of the properties from Definitions 15 and 16 has the same property.

It was shown in [34] that selective separability, R-separability and GN-separability are preserved by finite unions (to see that this is not immediate, it might be enough to mention that the question about H-separability remains open, and that SS^+ is not finitely additive [13]).

Proposition 30. A locally finite union of D -separable spaces is D -separable.

First we prove that this is the case for finite unions. The proof is a modification of the proof that selective separability is preserved by finite unions (see [34]).

Lemma 31. The union of two D -separable spaces is D -separable.

Proof. Let $X = A \cup B$ where A and B are D -separable and let $\{D_n : n \in \omega\} \subset X$ be a sequence of dense sets. Let $U_n = X \setminus ((\bigcup_{i \geq n} D_i) \cap A \cup \bigcup_{j < n} U_j)$ and $U = \bigcup_{n \in \omega} U_n$. Then the sets U_n are open in X and pairwise disjoint.

Claim 1. For each $i \geq n$ the set $D_i \cap B \cap U_n$ is dense in U_n .

Proof of Claim 1. Because U_n is open, D_i is dense in X , $X = A \cup B$, and for $i \geq n$, $D_i \cap U_n \cap A = \emptyset$. \triangle

Claim 2. There are discrete $G_n \subset D_n$ such that $\bigcup_{n \in \omega} G_n$ is dense in U .

Proof of Claim 2. By Proposition 29, part 1, the subspace $B \cap U_n$ is D -separable for every $n \in \omega$. Therefore there are discrete $G_i^n \subset D_i \cap B \cap U_n$ for every $i \geq n$ such that $\bigcup_{i \geq n} G_i^n$ is dense in $B \cap U_n$ and hence in U_n (because $B \cap U_n$ is dense in U_n). Let now $G_i = \bigcup_{n \leq i} G_i^n$. Since $G_i^n \subset U_n$ for every $n \leq i$ and $\{U_n : n \leq i\}$ is a family of pairwise disjoint open sets we have that G_i is a discrete subset of D_i . Moreover $\bigcup_{i \in \omega} G_i$ is dense in U_n for every $n \in \omega$, and hence in U . \triangle

Let now $V = X \setminus \overline{U}$. We claim that $(\bigcup_{i \geq n} D_i) \cap A$ is dense in V , and hence also in $A \cap V$. Indeed, if $x \in V$, then $x \notin \overline{U}$, so $x \notin U_n$ and $x \notin \overline{\bigcup_{j < n} U_j}$ for every $n \in \omega$, which together imply that $x \in (\bigcup_{i \geq n} D_i) \cap A$ for every $n \in \omega$.

Now $A \cap V$ is D -separable, so there are discrete $H_n \subset (\bigcup_{i \geq n} D_i) \cap A$ so that $\bigcup_{n \in \omega} H_n$ is dense in $V \cap A$ and hence in V .

For each $x \in H_n$ let $i_n(x) \in \omega \setminus n$ be such that $x \in D_{i_n(x)}$. Let $K_i = \{x : \exists n \in \omega (x \in H_n \text{ and } i_n(x) = i)\}$. Then K_i is a discrete subset of D_i and $\bigcup_{i \in \omega} K_i = \bigcup_{n \in \omega} H_n$, hence it is dense in V . Thus, if G_n is as in Claim 2, then $\bigcup_{n \in \omega} (G_n \cup K_n)$ is dense in X and each $G_n \cup K_n$ is discrete since $G_n \subset U$, $K_n \subset X \setminus \overline{U}$ and G_n and K_n are both discrete. So X is D-separable. \square

Proof of Proposition 30: First of all, by induction, Lemma 31 can be extended to any finite union.

Now, let $X = \bigcup \mathcal{Y}$ be a locally finite union, and let each $Y \in \mathcal{Y}$ be D-separable. For $n \geq 1$ put $X_n = \{x \in X : \text{there is a neighborhood } U \text{ of } x \text{ such that } |\{Y \in \mathcal{Y} : Y \cap U \neq \emptyset\}| \leq n\}$. Then the sets X_n are open in X , and $X = \bigcup_{n \in \omega} X_n$. Put $Z_1 = X_1$. For $n > 1$, put $Z_n = X_n \setminus \overline{X_{n-1}}$. Then the sets Z_n are open in X , pairwise disjoint, and $X = \bigcup_{n \geq 1} Z_n$. Further, each Z_n is a discrete union $Z = \bigsqcup \{Z_{n,A} : A \subset \mathcal{Y}, |A| = n\}$ where $Z_{n,A} = \{x \in Z_n : \text{there is a neighborhood } U \text{ of } x \text{ such that } |\{Y \in \mathcal{Y} : Y \cap U \neq \emptyset\}| = n\}$. Finally, $X = \bigsqcup \{Z_{n,A} : n \geq 1, A \subset \mathcal{Y}, |A| = n\}$ where the sets $Z_{n,A}$ are open in X and pairwise disjoint. By Lemma 31 (extended to arbitrary finite unions) each $Z_{n,A}$ is D-separable; hence by Proposition 29, part 3, so is $\bigsqcup \{Z_{n,A} : n \geq 1, A \subset \mathcal{Y}, |A| = n\}$, hence by part 2 of the same proposition so is X . \square

Question 32. Is every (locally) finite union of DH -separable spaces again DH -separable?

4.3. Products. The following two beautiful results witness how well-behaved d-separability is with respect to products.

Theorem 33. (Arhangel'skii, [5]) *Any product of d-separable spaces is d-separable.*

Theorem 34. (Juhász and Szentmiklóssy [40]) *For every T_1 -space X , $X^{d(X)}$ is d-separable.*

Theorems 33 and 34 imply two obvious corollaries:

Corollary 35. (1) *For every T_1 -space X there is $\kappa(X)$ such that for every $\kappa \geq \kappa(X)$, X^κ is d-separable.*

(2) *For every T_1 -space X there is a T_1 -space Z such that the product $X \times Z$ is d-separable. If X is Tychonoff, then so is Z .*

It is natural to ask if D-separability is (finitely or infinitely) productive and if the analogue of Corollary 35 is true for D-separability. It turns out that the product of two D-separable spaces does not have to be D-separable, and that for part 2 of Corollary 35 the answer is affirmative while for part 1 the situation is almost the opposite.

Theorem 36. *For every Tychonoff space X with $|X| > 1$ and every κ there is $\kappa' \geq \kappa$ such that $X^{\kappa'}$ is not D-separable.*

Theorem 37. *For every space X there is a Tychonoff space Z such that $X \times Z$ is DH^+ -separable.*

But before proving the above theorems let's examine the case of finite products.

Theorem 38. *Let X be a D-separable space (or has another property from Definitions 15 and 16) and Y be a space having a σ -disjoint π -base. Then $X \times Y$ is D-separable (or has the corresponding property).*

Proof. (For D-separability) Let $\mathcal{B} = \bigcup_{n < \omega} \mathcal{B}_n$ be a σ -disjoint π -base for Y and let $\{D_k : k < \omega\}$ be a countable sequence of dense subsets of $X \times Y$. Let $\{B_\alpha^n : \alpha < \tau_n\}$ enumerate \mathcal{B}_n and $\{A_n : n < \omega\}$ be a partition of ω . Observe that the set $\pi_X(D_k \cap \pi_Y^{-1}(B_\alpha^n))$ is dense in X for every $k \in A_n$ and for every $\alpha \in \tau_n$. Fix $\alpha < \tau_n$. Then for every $k \in A_n$ we can find a discrete set $E_k^\alpha \subset \pi_X(D_k \cap \pi_Y^{-1}(B_\alpha^n))$ such that $\bigcup_{k \in A_n} E_k^\alpha$ is dense in X . For every $x \in E_k^\alpha$ pick a point $f(x) \in \pi_X^{-1}(x)$ such that $\pi_Y(f(x)) \in B_\alpha^n$ and set $F_k^\alpha = \{f(x) : x \in E_k^\alpha\}$ and $F_k = \bigcup_{\alpha < \tau_k} F_k^\alpha$. Then F_k is discrete. In fact, let $(x, y) \in F_k$. Then $(x, y) \in F_k^\alpha$ for some $\alpha < \tau_k$ so $(x, y) = f(x)$ for some $y \in B_\alpha^n$. Moreover $x \in E_k^\alpha$. Now, since E_k^α is discrete in X there is an open $V \subset X$ such that $V \cap E_k^\alpha = \{x\}$. Finally, observe that, since \mathcal{B}_α^n is a disjoint family $(V \times B_\alpha^n) \cap F_k = \{(x, y)\}$, which shows that F_k is discrete. Moreover $F_k \subset D_k$ and $\bigcup_{k < \omega} F_k$ is dense in $X \times Y$, which proves that $X \times Y$ is D-separable.

The proofs for the other properties differ only by minor changes. \square

Example 39. [CH] *The product of two countable selectively separable spaces need not be D-separable.*

Proof: Let X be a selectively separable countable maximal regular crowded space such that X^2 has no dense slim set, see [34]. Let us check that the proof from [34] that X^2 is not selectively separable provides more: that X^2 is not D-separable. Enumerate $X = \{x_i : i \in \omega\}$. For every $n \in \omega$ let $D_n = \{(x, y) : x, y \notin \{x_i : i \leq n\}\}$. Then $\{D_i : i \in \omega\}$ is a sequence of dense sets in X . Let $E_n \subset D_n$ be a discrete set. Then $\bigcup_{n \in \omega} E_n$ meets every cross-section in a finite union of discrete sets. Now, in a crowded space, every discrete set is nowhere dense and finite unions of nowhere dense sets are nowhere dense. Therefore $\bigcup_{n \in \omega} E_n$ cannot be dense in X^2 , which proves that X^2 is not D-separable. \square

As a byproduct we get that under CH there exists a countable non-D-separable space. However, one can construct such an example even in ZFC.

Example 40. *There is a dense countable subset $X \subset 2^{\mathfrak{c}}$ such that X is not D-separable.*

Lemma 41. (1) *For every countable subset $S \subset 2^{\mathfrak{c}}$, there is $\alpha < \mathfrak{c}$ such that $\pi_{[0, \alpha)}|_S$ is a bijection.*

(2) *If a countable subset $S \subset 2^{\mathfrak{c}}$ is σ -discrete, then this can be witnessed by a projection to some initial face in $2^{\mathfrak{c}}$. That is, if $S = \bigcup_{n \in \omega} S_n$ where each S_n is countable and discrete, then there is $\alpha < \mathfrak{c}$ such that $\pi_{[0, \alpha)}(S_n)$ is discrete for each n , and $\pi_{[0, \alpha)}|_S$ is injective.*

Proof of lemma. (1) Pick countably many standard neighborhoods of points of S separating points of S and use the fact that $cf(\mathfrak{c}) > \omega$.

(2) Pick standard neighborhoods of points of S witnessing σ -discreteness. \square

Construction of Example 40: First, it is easy to construct pairwise disjoint dense countable subspaces Y_n , $n \in \omega$ in $2^{\mathfrak{c}}$ such that for every two distinct $y_1, y_2 \in Y = \bigcup_{n \in \omega} Y_n$, the set $I_{y_1, y_2} = \{\alpha < \mathfrak{c} : y_1(\alpha) \neq y_2(\alpha)\}$ has cardinality \mathfrak{c} . Using this, one can partition \mathfrak{c} as $\mathfrak{c} = \bigcup \{C_A : A \subset Y\}$ so that each C_A has cardinality \mathfrak{c} , and for every two distinct $y_1, y_2 \in Y$, $C_{\{y_1, y_2\}} \subset I_{y_1, y_2}$ (*).

Next, by induction on $0 \leq \alpha < \mathfrak{c}$, we will construct countable subspaces $Z_\alpha \subset 2^\mathfrak{c}$ that will take the form $Z_\alpha = \{y_\alpha : y \in Y\}$. We also denote $Z_{\alpha,n} = \{y_\alpha : y \in Y_n\}$, so $Z_\alpha = \bigcup_{n \in \omega} Z_{\alpha,n}$. The points of Z_α s are going to have the following property: if $0 \leq \gamma \leq \alpha \leq \beta < \mathfrak{c}$ then for all $y \in Y$, $y_\beta(\gamma) = y_\alpha(\gamma)$ (**).

To start the induction, we set $Z_0 = Y$, that is $y_0 = y$ for all $y \in Y$.

Now let $0 < \alpha < \mathfrak{c}$, and suppose X_γ s have been defined for all $\gamma < \alpha$. Let $y \in Y$. To define the corresponding point $y_\alpha \in Z_\alpha$, we have to define $y_\alpha(\gamma)$ for all γ , $0 \leq \gamma < \mathfrak{c}$. If $0 \leq \gamma < \alpha$, then we set $y_\alpha(\gamma) = y_\gamma(\gamma)$ (and thus condition (**) continues to hold.)

To define $y_\alpha(\alpha)$ we need some auxiliary notation. By the previous, we have in fact defined $\pi_{[0,\alpha)}(y_\alpha)$ for all $y \in Y$. For a subset $B \subset Y$, set $B_{<\alpha} = \{\pi_{[0,\alpha)}(y_\alpha) : y \in B\} \subset 2^{[0,\alpha)}$. We have $\alpha \in C_A$ for some $A \subset Y$. If all the following conditions hold:

- (1) A is infinite,
- (2) the mapping $A \rightarrow A_{<\alpha}$ given by $y \mapsto \pi_{[0,\alpha)}(y_\alpha)$ is a bijection,
- (3) for every $n \in \omega$, $(A \cap Y_n)_{<\alpha}$ is discrete,

then we set $y_\alpha(\alpha) = 0$ for all $y \in A$. Otherwise we set $y_\alpha(\alpha) = y(\alpha)$.

Finally, for all γ with $\alpha < \gamma < \mathfrak{c}$, we set $y_\alpha(\gamma) = y(\gamma)$. This concludes the construction of Z_α .

Now we define the countable subspace $X \subset 2^\mathfrak{c}$, $X = \{\tilde{y} : y \in Y\}$ by setting $\tilde{y}(\alpha) = y_\alpha(\alpha)$ for all $y \in Y$. It follows from (**) that $\tilde{y}(\gamma) = y_\alpha(\gamma)$ whenever $0 \leq \gamma \leq \alpha < \mathfrak{c}$. For $n \in \omega$, we set $X_n = \{\tilde{y} : y \in Y_n\}$, thus we have $X = \bigcup_{n \in \omega} X_n$.

Claim 1. The mapping $y \mapsto \tilde{y}$ from Y onto X is a bijection.

Proof of Claim 1. Indeed, if y_1, y_2 be distinct elements of Y , then by our construction, since by (*) $C_{\{y_1, y_2\}} \subset I_{y_1, y_2}$, we have $\tilde{y}_1(\alpha) = y_1(\alpha) \neq y_2(\alpha) = \tilde{y}_2(\alpha)$ for every $\alpha \in C_{\{y_1, y_2\}}$. \triangle

Claim 2. Each X_n is dense in $2^\mathfrak{c}$ (and thus in X).

Proof of Claim 2. Indeed, let $F \subset \mathfrak{c}$ be finite, and let $\varphi \in 2^F$. We have to find $\tilde{y} \in X_n$ such that $\tilde{y}|_F = \varphi$. For each $i \in F$, there is $A_i \subset Y$ such that $i \in C_{A_i}$. Put $\mathcal{A} = \{A_i : i \in F \text{ and conditions (1), (2), (3) were satisfied when the } i\text{th coordinates of the points of } X_i \text{ were defined}\}$. Pick α^* with $\max(F) < \alpha^* < \mathfrak{c}$. Using Lemma 41, (1), we can assume that $\pi_{[0,\alpha^*)}|_X$ is a bijection. Then $T = \pi_{[0,\alpha^*)}(\bigcup\{A_i \cap Y_n : A_i \in \mathcal{A}\})$ is a finite union of discrete subspaces of $2^{[0,\alpha^*)}$, and thus T is nowhere dense in $2^{[0,\alpha^*)}$. So $T' = \pi_{[0,\alpha^*)}(Y_n) \setminus T$ is dense in $2^{[0,\alpha^*)}$. Pick $t \in T'$ with $t|_F = \varphi$ and $y \in Y_n$ with $\pi_{[0,\alpha^*)}(y) = t$. Then $\tilde{y} \in X_n$, and $\tilde{y}|_F = y|_F = t|_F = \varphi$. \triangle

Claim 3. For any choice of discrete $S_n \subset X_n$, $n \in \omega$, the set $S = \bigcup_{n \in \omega} S_n$ is not dense in $2^\mathfrak{c}$ (and thus not dense in X).

Proof of Claim 3. By Lemma 41, (2), there is $\alpha^* < \mathfrak{c}$ such that $\pi_{[0,\alpha^*)}(S_n)$ is discrete for each n , and $\pi_{[0,\alpha^*)}|_S$ is injective. Put $A = \{y \in Y : \tilde{y} \in S\}$. Pick $\alpha^{**} \in C_A$ so that $\alpha^{**} \geq \alpha^*$. Then $\tilde{y}(\alpha^{**}) = 0$ for every $\tilde{y} \in S$, and thus S is not dense in $2^\mathfrak{c}$. \triangle

Claims 2 and 3 show that X is as was desired. \square

Corollary 42. $2^\mathfrak{c}$ is not D -separable.

Question 43. What is $\mathfrak{cds} = \min\{\tau : 2^\tau \text{ contains a dense countable subspace which is not D-separable}\}$?

Question 44. What is $\mathfrak{ds} = \min\{\tau : 2^\tau \text{ is not D-separable}\}$?

Question 45. Is $\mathfrak{cds} = \mathfrak{ds}$?

Question 46. Is it true that for every separable Tychonoff space X there is a separable Tychonoff space Y such that $X \times Y$ is D-separable?

4.3.1. *Proof of Theorem 36.* It suffices to show that (\dagger) for every Tychonoff X there is τ such that X^τ is not D-separable. Indeed, applying (\dagger) to $X' = X^\kappa$ we get the original statement of the theorem. In the case of a finite X , $\tau = \mathfrak{c}$ works by an easy modification of the argument from Example 40, so we assume $\lambda = |X|$ is infinite. Next, if D is dense in X and D^τ is not D-separable, then neither is X^τ . So we can pass from X to a dense subspace of minimal cardinality and thus assume $|X| = d(X)$ when proving the following

Theorem 47. *For every Tychonoff X , $X^{2^{d(X)}}$ is not D-separable.*

Proof. The argument is parallel to one from Example 40, so we will omit some details. Fix a point $x_0 \in X$.

Let $\tau = 2^\lambda$ (where $\lambda = |X| = d(X)$). Since $\text{cf}(\tau) > \lambda$ we get the following:

Lemma 48. (1) *For every subset $S \subset X^\tau$, such that $|S| \leq \lambda$ there is $\alpha < \tau$ such that $\pi_{[0,\alpha)}|_S$ is a bijection.*

(2) *If a subset $S \subset 2^\tau$ such that $|S| \leq \lambda$ is σ -discrete, then this can be witnessed by a projection to some initial face in X^τ . That is, if $S = \bigcup_{n \in \omega} S_n$ where each S_n is discrete, then there is $\alpha < \tau$ such that $\pi_{[0,\alpha)}(S_n)$ is discrete for each n , and $\pi_{[0,\alpha)}|_S$ is injective.*

The routine proof of the next lemma is omitted.

Lemma 49. *There exist pairwise disjoint dense subspaces Y_n , $n \in \omega$ in X^τ such that $|Y_n| \leq \lambda$ and for every two distinct $y_1, y_2 \in Y = \bigcup_{n \in \omega} Y_n$, the set $I_{y_1, y_2} = \{\alpha < \tau : y_1(\alpha) \neq y_2(\alpha)\}$ has cardinality τ .*

Using this, one can partition τ as $\tau = \bigcup \{C_A : A \subset Y\}$ so that each C_A has cardinality τ , and for every two distinct $y_1, y_2 \in Y$, $C_{\{y_1, y_2\}} \subset I_{y_1, y_2}$ (*).

Next, by induction on $0 \leq \alpha < \tau$, we will construct λ -sized subspaces $Z_\alpha \subset X^\tau$ that will take the form $Z_\alpha = \{y_\alpha : y \in Y\}$. We also denote $Z_{\alpha, n} = \{y_\alpha : y \in Y_n\}$, so $Z_\alpha = \bigcup_{n \in \omega} Z_{\alpha, n}$. The points of Z_α s are going to have the following property: if $0 \leq \gamma \leq \alpha \leq \beta < \tau$ then for all $y \in Y$, $y_\beta(\gamma) = y_\alpha(\gamma)$ (**).

To start the induction, we set $Z_0 = Y$, that is $y_0 = y$ for all $y \in Y$.

Now let $0 < \alpha < \tau$, and suppose X_γ s have been defined for all $\gamma < \alpha$. Let $y \in Y$. To define the corresponding point $y_\alpha \in Z_\alpha$, we have to define $y_\alpha(\gamma)$ for all γ , $0 \leq \gamma < \tau$. If $0 \leq \gamma < \alpha$, then we set $y_\alpha(\gamma) = y_\gamma(\gamma)$ (and thus condition (**) continues to hold.)

To define $y_\alpha(\alpha)$ we need some auxiliary notation. By the previous, we have in fact defined $\pi_{[0,\alpha)}(y_\alpha)$ for all $y \in Y$. For a subset $B \subset Y$, set $B_{<\alpha} = \{\pi_{[0,\alpha)}(y_\alpha) : y \in B\} \subset X^{[0,\alpha)}$. We have $\alpha \in C_A$ for some $A \subset Y$. If all the following conditions hold:

- (1) A is infinite,

- (2) the mapping $A \rightarrow A_{<\alpha}$ given by $y \mapsto \pi_{[0,\alpha)}(y_\alpha)$ is a bijection,
- (3) for every $n \in \omega$, $(A \cap Y_n)_{<\alpha}$ is discrete,

then we set $y_\alpha(\alpha) = x_0$ for all $y \in A$. Otherwise we set $y_\alpha(\alpha) = y(\alpha)$.

Finally, for all γ with $\alpha < \gamma < \tau$, we set $y_\alpha(\gamma) = y(\gamma)$. This concludes the construction of Z_α .

So we have Z_α satisfying (**) for all $\alpha < \tau$. Now we define the subspace $\tilde{Y} \subset X^\tau$ by $\tilde{Y} = \{\tilde{y} : y \in Y\}$ where $\tilde{y}(\alpha) = y_\alpha(\alpha)$ for all $y \in Y$. It follows from (**) that $\tilde{y}(\gamma) = y_\alpha(\gamma)$ whenever $0 \leq \gamma \leq \alpha < \tau$. For $n \in \omega$, we set $\tilde{Y}_n = \{\tilde{y} : y \in Y_n\}$, thus we have $\tilde{Y} = \bigcup_{n \in \omega} \tilde{Y}_n$.

Claim 1. The mapping $y \mapsto \tilde{y}$ from Y onto \tilde{Y} is a bijection.

Proof of Claim 1. Indeed, if y_1, y_2 be distinct elements of Y , then by our construction, since by (*) $C_{\{y_1, y_2\}} \subset I_{y_1, y_2}$, we have $\tilde{y}_1(\alpha) = y_1(\alpha) \neq y_2(\alpha) = \tilde{y}_2(\alpha)$ for every $\alpha \in C_{\{y_1, y_2\}}$. \triangle

Claim 2. Each \tilde{Y}_n is dense in X^τ (and thus in \tilde{Y}).

Proof of Claim 2. Indeed, let $F \subset \tau$ be finite, and let $\varphi \in (\mathcal{T} \setminus \{\emptyset\})^F$ (where \mathcal{T} is the topology of X). We have to find $\tilde{y} \in \tilde{Y}_n$ such that (+) $\tilde{y}(i) \in \varphi(i)$ for every $i \in F$. For each $i \in F$, there is $A_i \subset Y$ such that $i \in C_{A_i}$. Put $\mathcal{A} = \{A_i : i \in F \text{ and conditions (1), (2), (3) were satisfied when the } i\text{th coordinates of the points of } X_i \text{ were defined}\}$. Pick α^* with $\max(F) < \alpha^* < \tau$. Using Lemma 48, (1), we can assume that $\pi_{[0, \alpha^*)}|_{\tilde{Y}}$ is a bijection. Then $T = \pi_{[0, \alpha^*)}(\bigcup\{A_i \cap Y_n : A_i \in \mathcal{A}\})$ is a finite union of discrete subspaces of $X^{[0, \alpha^*)}$, and thus T is nowhere dense in $X^{[0, \alpha^*)}$. So $T' = \pi_{[0, \alpha^*)}(Y_n) \setminus T$ is dense in $X^{[0, \alpha^*)}$. Pick $t \in T'$ with $t(i) \in \varphi(i)$ for every $i \in F$. There is $y \in Y_n$ with $\pi_{[0, \alpha^*)}(y) = t$. Then $\tilde{y} \in \tilde{Y}_n$, and $\tilde{y}|_F = y|_F$, so \tilde{y} satisfies (+). \triangle

Claim 3. For any choice of discrete $S_n \subset \tilde{Y}_n$, $n \in \omega$, the set $S = \bigcup_{n \in \omega} S_n$ is not dense in X^τ (and thus not dense in X).

Proof of Claim 3. By Lemma 48, (2), there is $\alpha^* < \tau$ such that $\pi_{[0, \alpha^*)}(S_n)$ is discrete for each n , and $\pi_{[0, \alpha^*)}|_S$ is injective. Put $A = \{y \in Y : \tilde{y} \in S\}$. Pick $\alpha^{**} \in C_A$ so that $\alpha^{**} \geq \alpha^*$. Then $\tilde{y}(\alpha^{**}) = x_0$ for every $\tilde{y} \in S$, and thus S is not dense in X^τ . \triangle

Claims 2 and 3 show that \tilde{Y} is not D-separable. Since \tilde{Y} is dense in X^τ it follows that X^τ is not D-separable. \square

Question 50. Is it true that for every Tychonoff space X there is κ such that for all $\kappa' \geq \kappa$, $X^{\kappa'}$ is not D-separable?

4.3.2. *Proof of Theorem 37.* More specifically, we will prove:

Theorem 51. *Let X be any space, and let Y be any space such that $\pi w(X) \leq \pi w(Y) = \kappa$ and Y contains a cellular family of size κ . Then $X \times Y^\omega$ is DH^+ -separable.*

(Then, for Theorem 37, one can take $Z = Y^\omega$. As Y , one can take the discrete space of size $\pi w(X)$ or a one-point compactification of such a space, so Z in Theorem 37 can be in addition assumed compact.)

Proof. Let \mathcal{U} and \mathcal{V} be π -bases of X and Y having minimal size. Let $\{C_\alpha : \alpha < \kappa\}$ be a cellular family in Y . For $m \in \omega$, let $\{e_\alpha^m : \alpha < \kappa\}$ be an enumeration of $\mathcal{U} \times \mathcal{V}^m$.

On the m th move, One chooses a dense subspace $S_m \subset X \times Y^\omega$, and Two, for every $m \in \omega$ and $\alpha < \kappa$ selects $d_\alpha^m \in (\prod_{n \in \omega} W_n^{\alpha, m}) \cap S_m$ where $W_n^{\alpha, m} = e_\alpha^m(n)$ for $n \leq m$, $W_{m+1}^{\alpha, m} = C_\alpha$ and $W_n^{\alpha, m} = Y$ for $n > m+1$. Let $D_m = \{d_\alpha^m : \alpha < \kappa\}$. Then $D_m \subset S_m$, D_m is discrete, and D_m intersects every non-empty open set in $X \times Y^\omega$ that depends only on the first $m+1$ coordinates. Thus every non-empty open set in $X \times Y^\omega$ intersects all but finitely many D_m s. \square

A consequence of the above Theorem is that there is no single cardinal κ such that X^κ is not D -separable for every space X .

Corollary 52. *For every κ , and every $\lambda \leq \kappa$, $(D(\kappa))^\lambda$ is D -separable (where $D(\kappa)$ is the discrete space of cardinality κ).*

Proof. This is trivial if κ is finite. So assume κ is infinite. Now set $X = Y = D(\kappa)^\omega$ in Theorem 51 and observe that $(D(\kappa)^\kappa)^\omega$ and $D(\kappa)^\kappa$ are homeomorphic. \square

Another notable consequence of Theorem 51 is the fact that the ω -power of any linearly ordered space is D -separable. This follows from the following result of Petr Simon.

Lemma 53. [56] *Let X be a linearly ordered topological space. Then X^2 contains a cellular family of size $d(X)$.*

Corollary 54. *Let X be a linearly ordered topological space. Then X^ω is DH^+ -separable.*

Proof. Since $\pi w(X) = d(X)$ in linearly ordered spaces X^2 contains a pairwise disjoint open family of size $\pi w(X)$. Now let $Y = X^2$ in Theorem 51. \square

So, although a Suslin Line is not even d -separable, its ω -power is DH^+ -separable.

4.4. Some more open problems. Tkachuk presented a large collection of sufficient conditions and necessary conditions of d -separability of $C_p(X)$ in [60]. Tkachuk gave a CH example of a compact space X with a non- d -separable $C_p(X)$ and asked for a ZFC example of a Tychonoff space or even a compact space X with non- d -separable $C_p(X)$. A Tychonoff ZFC example was presented in [40].

Problem 55. (1) Characterize X such that $C_p(X)$ is D -separable.

(2) More specifically, suppose $C_p(X)$ is d -separable. Under what additional conditions on X is $C_p(X)$ D -separable?

Recall that a compact space X is selectively separable iff X has a countable π -base ([16]). This is a consequence of the fact that a compact space X has a countable π -base iff every dense subspace of X is separable [38].

Let $dd(Y)$ be the least cardinal κ such that Y has a dense set which is the union of κ many discrete sets. Let $d\delta(X) = \sup\{dd(D) : D \text{ is dense in } X\}$. Let $d\pi(X)$ be the least cardinal κ such that X has a π -base which is the union of κ many disjoint collections.

Conjecture 56.

(1) *A compact space X is D -separable iff X has a σ -disjoint π -base.*

(2) *Let X be a compact space. Then $d\delta(X) = d\pi(X)$.*

By Theorem 38 if Conjecture 56, 1 is true then the answer to the following question is positive.

Question 57. Is the product of two compact D-separable spaces still D-separable?

Recall that a space is called an L -space if it is hereditarily Lindelöf but not separable. Tkachuk [60] constructed under CH an L -space X such that X^2 is d -separable. Later on, Moore [47] showed that a slight modification of his ZFC example of an L -space provides a ZFC example of an L -space with a d -separable square.

Question 58. Is there a non-D-separable space X such that X^2 is D-separable? Is there even a non- d -separable space with this property?

Question 59. Is there (in any model of ZFC) an example of an L -space with a D-separable square?

Note that replacing D-separability with selective separability both questions have easily a negative answer.

Also, the influence of convergence properties on D-separability is not clear yet.

Question 60.

- (1) Is every Fréchet d -separable space D -separable? What about $\Sigma(2^\kappa)$?
- (2) Is there a sequential d -separable (separable, countable) non-D-separable space?
- (3) Is there a Whyburn d -separable (separable, countable) non-D-separable space?

5. MORE ON MAXIMAL (AND SUBMAXIMAL) SPACES

We conclude with some remarks on the interesting case of maximal and submaximal spaces. In a submaximal space every dense set is open, so in some sense dense sets are “big”. This implies “a lot of freedom” in choosing a finite set and this in turn could suggest that a maximal space can easily be selectively separable, but we will see below that often things go differently.

In [16] it was shown that assuming $\mathfrak{d} = \omega_1$ there is a maximal regular space which is not selectively separable, and it was asked (1) whether or not such an example is possible within ZFC, and (2) is it true (at least consistently) that every countable maximal regular space is not selectively separable? Here is the progress obtained since then:

Theorem 61.

- (1) (Barman and Dow, [12]) *There is (within ZFC) a countable maximal regular space which is not selectively separable.*
- (2) (Barman and Dow, [12], Repovš and Zdomskyy, [50]) *Consistently, there is no submaximal SS space (specifically, the existence of such a space implies the existence of a separable P -set in ω^* while the existence of a ccc P -set in ω^* is known to be independent from ZFC from [30]).²*

²The result is stated in [12], [50] only for maximal spaces, but it is easy to notice that the argument uses only submaximality.

- (3) (Barman and Dow, [12]) [MA_{ctble}] *There exists a maximal regular countable selectively separable space. (So the existence of a maximal regular selectively separable space is independent of ZFC.)*
- (4) (Gruenhage and Sakai, [34]) [CH] *There is a maximal space X such that X is R -separable but X^2 is not selectively separable.*³
- (5) (Barman and Dow, [12]) *Every crowded SS^+ space is resolvable. Hence no maximal space is SS^+ .*⁴

Many results on maximal regular spaces, in particular the construction of Barman and Dow (Theorem 61, part 1 above) are based on the following tool found by van Douwen:

Theorem 62. [25] *For any countable regular crowded space (X, τ) there is a stronger regular topology σ such that the space (X, σ) has a dense subset which is a maximal space.*

Below we present an alternative proof of Theorem 61, part 1 (based on the space $Seq(\mathcal{F})$ as the starting point), and construct a maximal regular countable SS space using a weaker assumption than in Theorem 61, part 3, namely $\mathfrak{d} = \mathfrak{c}$. Then we discuss maximal D -separable spaces.

Theorem 63. *There exists a countable regular maximal space which is not selectively separable.*

Proof. Start by letting $(X, \tau) = Seq(\mathcal{F})$ and fix the sequence of dense subsets $\{H_n : n < \omega\}$, where $H_n = \bigcup \{^k\omega : n \leq k < \omega\}$.

Step 1: use van Douwen's theorem to find $\sigma \supset \tau$ and a dense subset Z of (X, σ) which is a regular maximal space.

Step 2: since each H_n has a closed scattered complement in (X, τ) , it follows that H_n remains dense and open in (X, σ) and so the set $D_n = Z \cap H_n$ is dense in Z .

Step 3: the sequence $\{D_n : n < \omega\}$ cannot have a "good selection" because it would be also a "good selection" for the sequence $\{H_n : n < \omega\}$ in (X, τ) . \square

Recall that first Gruenhage under [CH] (later included in [34]) and then Barman and Dow under MA_{ctble} have shown the existence of a countable regular maximal selectively separable space. Gruenhage's construction gives a stronger result: a maximal R -separable space whose square is not selectively separable. We are going to show that, with respect to the weaker task to have just a maximal selectively separable space, $\mathfrak{d} = \mathfrak{c}$ suffices. The construction we present below follows the pattern of that of Gruenhage.

Lemma 64. *Let X be a space and $x \in X$. If $t(x, X) = \omega$ and $\chi(x, X) < \mathfrak{d}$, then X has countable fan tightness at x .*

Proof. Let $\{A_n : n < \omega\}$ be a sequence of sets such that $x \in \overline{A_n}$ for each n . Since $t(x, X) = \omega$, we may assume each A_n countable and write $A_n = \{a_{n,k} : k < \omega\}$. Let $\{U_\alpha : \alpha < \kappa\}$ be a local base at x with $\kappa < \mathfrak{d}$. For any α we may define a function $f_\alpha \in {}^\omega\omega$ by letting $f_\alpha(n) = \min\{k : a_{n,k} \in U_\alpha \cap A_n\}$. Since $\kappa < \mathfrak{d}$, the family $\{f_\alpha : \alpha < \kappa\}$ cannot be dominating and so there exists $g \in {}^\omega\omega$ such that the set

³The first author and Gruenhage obtained the same result under a weaker assumption MA_{ctble} .

⁴And one can see from the argument in [12] that, more generally, no crowded submaximal space) is SS^+ .

$\{n : f_\alpha(n) \leq g(n)\}$ is infinite for each α . Now, by letting $F_n = \{a_{n,k} : k \leq g(n)\}$, we may easily check that $x \in \overline{\bigcup\{F_n : n < \omega\}}$. \square

In [19] it is shown that any crowded space of countable fan tightness is ω -resolvable. So we have:

Corollary 65. *A countable crowded space of weight less than \mathfrak{d} is ω -resolvable.*

The above corollary is the main ingredient in the proof of the following:

Lemma 66. $[\mathfrak{d} = \mathfrak{c}]$ *Let (X, τ) be a countable crowded regular space of weight $\leq \kappa$ where $\kappa < \mathfrak{c}$. Then:*

(1) *If A is a dense subset of (X, τ) , then there is an enlargement σ_1 of τ such that (X, σ_1) is a regular crowded space of weight $\leq \kappa$, $A \in \sigma_1$ and each dense open set in (X, τ) remains dense in (X, σ_1) ;*

(2) *If A is a crowded subset in (X, τ) , then there exists an enlargement σ_2 of τ such that (X, σ_2) is a regular crowded space of weight $\leq \kappa$, A is either open in (X, σ_2) or it has an isolated point in (X, σ_2) and each dense open set in (X, τ) remains dense in (X, σ_2) .*

Proof. Part 1: by Corollary 65 the subspace A is ω -resolvable and we may write $A = \bigcup\{A_n : n < \omega\}$, where each A_n is dense and $A_i \cap A_j = \emptyset$ whenever $i \neq j$. σ_1 is the topology on X generated by $\tau \cup \{A_n : n < \omega\} \cup \{X \setminus A_n : n < \omega\}$.

Part 2: if A is dense in X , then we may argue as in part 1. If not, let $V = \text{Int}(X \setminus A)$ and consider the topology τ' generated by $\tau \cup \{\overline{V}\}$. τ' is regular and any dense open set in τ remains dense in τ' . If $\overline{V} \cap A = \emptyset$, then apply part 1 to the space (X, τ') and the dense set $\overline{V} \cup A$. In the resulting topology σ_2 the set A is open. If $\overline{V} \cap A \neq \emptyset$, then pick a point $p \in \overline{V} \cap A$ and apply part 1 to the space (X, τ') and the dense set $X \setminus (\overline{V} \cap A \setminus \{p\})$. In the resulting topology σ_2 the point p is isolated in A . \square

Theorem 67. $[\mathfrak{d} = \mathfrak{c}]$ *There exists a countable regular maximal selectively separable space.*

Proof. Let τ_0 be a regular crowded second countable topology on the set ω . List all infinite subsets of ω as $\{A_\alpha : \alpha < \mathfrak{c}\}$ and all ω -sequences of subsets of ω as $\{\langle D_n^\alpha : n < \omega \rangle : \alpha < \mathfrak{c}\}$ (in the latter each element is listed \mathfrak{c} -many times). For any $\alpha < \mathfrak{c}$ we will construct a crowded regular topology τ_α on ω in such a way that:

- (1) if $\beta < \alpha$ then $\tau_\beta \subseteq \tau_\alpha$ and any dense open set in τ_β remains dense in τ_α ;
- (2) the weight of τ_α is at most $|\alpha| + \omega$;
- (3) if A_α is dense in τ_α , then A_α is dense open in $\tau_{\alpha+1}$;
- (4) if A_α is not dense but crowded in τ_α , then either A_α is open in $\tau_{\alpha+1}$ or A_α has an isolated point in $\tau_{\alpha+1}$;
- (5) if $\langle D_n^\alpha : n < \omega \rangle$ is a sequence of dense open sets in τ_α , then there are finite sets $F_n^\alpha \subseteq D_n^\alpha$ such that the set $\bigcup\{F_n^\alpha : n < \omega\}$ is dense open in $\tau_{\alpha+1}$.

Suppose to have already defined topologies τ_β and sequences $\langle F_n^\beta : n < \omega \rangle$ for $\beta < \alpha$ satisfying the above conditions. If α is a limit ordinal, then we take as τ_α the topology generated by $\bigcup\{\tau_\beta : \beta < \alpha\}$. In this case, only condition 2 needs to be checked. Now, assume $\alpha = \gamma + 1$. If A_γ is crowded, then apply Lemma 66 to get a topology τ' (τ' is either σ_1 or σ_2 from Lemma 66 according to the fact that A_γ is or is not dense in τ_γ). Next, if $\langle D_n^\gamma : n < \omega \rangle$ is a sequence of dense open

sets in τ_γ (and so even in τ'), we may use Proposition 7, Part A to find finite sets $F_n^\gamma \subseteq D_n^\gamma$ in such a way that the set $B = \bigcup \{F_n^\gamma : n < \omega\}$ is dense in τ' . To finish the construction, apply again part 1 of Lemma 66 to get a topology $\tau_{\gamma+1}$ which is the enlargement of τ' where B is dense open.

Let τ be the topology generated by $\bigcup \{\tau_\alpha : \alpha < \mathfrak{c}\}$. If the set A is crowded in τ and $A = A_\alpha$, then A is also crowded in τ_α . By construction, $A = A_\alpha$ is open in $\tau_{\alpha+1}$ and so even in τ (the second possibility in condition 4 cannot occur because A cannot have isolated points in $\tau_{\alpha+1}$). The fact that every crowded subset of τ is open ensures that τ is a maximal topology [25]. If $\langle D_n : n < \omega \rangle$ is a sequence of dense sets in τ , then each D_n is dense in each τ_α and so there is some $\beta < \mathfrak{c}$ such that each D_n is dense open in τ_β . Since every ω -sequence of subsets of ω is listed \mathfrak{c} -many times, there is an ordinal $\gamma \geq \beta$ such that $\langle D_n : n < \omega \rangle = \langle D_n^\gamma : n < \omega \rangle$. By condition 5 we get finite sets $F_n^\gamma \subseteq D_n^\gamma = D_n$ in such a way that the set $\bigcup \{F_n^\gamma : n < \gamma\}$ is dense open in $\tau_{\gamma+1}$ and so dense even in τ . This shows that the space (ω, τ) is selectively separable. \square

Now we go back to D-separability. It turns out that, at least in the countable case, maximal spaces are always D-separable.

Theorem 68. *Let X be submaximal. Then the following conditions are equivalent:*

- (1) X is D-separable;
- (2) X is d-separable;
- (3) X is σ -discrete;
- (4) X is σ -closed discrete.

The following is immediate:

Corollary 69. *A countable submaximal space is D-separable.*

Proof of Theorem 68. First, we prove the theorem for the special case when X is crowded. The implications (1) \Rightarrow (2) and (4) \Rightarrow (3) \Rightarrow (2) are obvious.

(2) \Rightarrow (1)&(4): Let $H = \bigcup_{n \in \omega} H_n$ be a dense subspace of X where each H_n is discrete. Further, let $\langle D_n : n \in \omega \rangle$ be an arbitrary sequence of dense subsets of X . For $n \in \omega$, put $D'_n = D_n \cap (H \setminus H_n)$. Then D'_n is dense in X . Next, put $G_n = (D'_0 \cap \dots \cap D'_n) \setminus D'_{n+1}$. Each G_n is closed discrete being a subset of the complement to the dense set D'_{n+1} . By construction, we have $G_n \subseteq D_n$ and $D'_0 = \bigcup_{n \in \omega} G_n$. Therefore, $\bigcup_{n \in \omega} G_n$ is dense in X . Last, put $G_\omega = X \setminus D'_0$. Then $X = \bigcup_{n \leq \omega} G_n$ is a countable union of closed discrete subspaces.

So we have proved the theorem for crowded X . It follows in particular that (*) every countable crowded submaximal space is D-separable. Now let X be arbitrary submaximal space. Replace every isolated point of X with a copy of a countable crowded regular maximal space. Call the resulting space \tilde{X} ; \tilde{X} is crowded. It is easy to deduce from (*) that X has one of the properties (1) through (4) iff so does \tilde{X} . This completes the proof. \square

We will see that submaximal spaces can never be D^+ -separable (see Theorem 76). However, we don't know the answer to the following question.

Question 70. Is there a countable submaximal space which is not DH -separable?

Arhangel'skii and Collins asked in [8] if all submaximal spaces are σ -discrete. Schröder proved in [54] that assuming $V=L$ the answer is affirmative. Thus we get:

Corollary 71. $[V=L]$ *Every submaximal space is D -separable.*

On the other hand, Kunen, Szymanski and Tall showed [41] that the existence of a measurable cardinal is consistent with ZFC iff the existence of a Tychonoff crowded SIB space is consistent with ZFC. Further, Levy and Porter proved the following:

Proposition 72. ([43], Proposition 3.1 and a remark after it) *The following conditions are equivalent:*⁵

- (1) *There exists a submaximal Hausdorff space which is not σ -discrete;*
- (2) *There exists a crowded submaximal Hausdorff space which is not σ -discrete;*
- (3) *There exists a maximal space which is not σ -discrete;*
- (4) *There exists a crowded submaximal Hausdorff space which is not strongly σ -discrete;*
- (5) *There exists a maximal space which is not strongly σ -discrete;*
- (6) *There exists a crowded SIB space;*
- (7) *There exists a crowded Hausdorff space X such that every real-valued function defined on X is continuous at some point.*

It follows [43] that the existence of a crowded submaximal (or, equivalently, the existence of a maximal) Hausdorff space which is not σ -discrete is equiconsistent with a measurable cardinal. We see from Theorem 68 that the existence of a submaximal space which is not D -separable can be added to the list of conditions in Proposition 72. Therefore we get:

Corollary 73. *The existence of a submaximal space which is not D -separable is equiconsistent with a measurable cardinal.*

As there exist (in ZFC) countable regular maximal non-selectively separable spaces, Corollary 69 implies the existence of a space with some of the properties of Example 25.

Corollary 74. *There exists a countable regular (maximal) D -separable non-selectively separable space.*

Unlike Example 25 such a space can never be sequential. Indeed, maximal spaces contain no non-trivial convergent sequences. However, Malykhin [45] has shown that maximal spaces can carry a group structure, so we wonder if Corollary 74 can be improved in the following way:

Question 75. *Is there a countable regular maximal non-selectively separable topological group?*

Such an improvement can never be achieved in ZFC alone because Protasov [49] has shown that there are models of ZFC with no maximal topological groups.

We conclude by showing that a crowded submaximal space cannot be D^+ -separable. It was shown in [12] that every crowded SS^+ space is resolvable (and hence cannot be submaximal). We note that the argument extends to D^+ -separable spaces.

Theorem 76. *Every crowded D^+ -separable space is ω -resolvable (and hence non-submaximal).*

⁵The equivalence of conditions (6) and (7) is attributed in [43] to Malykhin [46].

Proof. Let σ be the winning strategy for Two in the game $G_{\text{dis}}(\mathcal{D}, \mathcal{D})$ on the space X . It suffices to show that any dense $D \subset X$ contains two disjoint dense subsets. Let $S_0 = \sigma(D)$, $T_0 = \sigma(D \setminus S_0)$, $S_1 = \sigma(D, D \setminus (S_0 \cup T_0))$, $T_1 = \sigma(D, D \setminus (S_0 \cup T_0 \cup S_1))$, $S_2 = \sigma(D, D \setminus (S_0 \cup T_0), D \setminus (S_0 \cup T_0 \cup S_1 \cup T_1))$, etc... Because σ is winning the disjoint sets $\bigcup_{n \in \omega} S_n$ and $\bigcup_{n \in \omega} T_n$ are dense. \square

Proposition 77. *If X is D -separable and ω -resolvable, then there is a family \mathcal{G} of dense subspaces of X such that $|\mathcal{G}| = \mathfrak{c}$ and $G \cap G'$ is nowhere dense, for every distinct $G, G' \in \mathcal{G}$.*

Proof. Fix a family $\{Y_n : n \in \omega\}$ of pairwise disjoint dense subsets of X . Also fix an almost disjoint family \mathcal{A} of infinite subsets of ω such that $|\mathcal{A}| = \mathfrak{c}$. For each $A \in \mathcal{A}$ apply the definition of D -separability to the family of dense subspaces $\{Y_n : n \in A\}$ to get discrete $F_{A,n} \subset Y_n$ such that $Z_A = \bigcup_{n \in A} F_{A,n}$ is dense in X . It remains to note that whenever $A, A' \in \mathcal{A}$ are distinct, $Z_A \cap Z_{A'}$ is the union of finitely many discrete sets and thus nowhere dense. \square

Corollary 78. *If X is D -separable, ω -resolvable and $\Delta(X) < \mathfrak{c}$ (in particular, if X is countable), then X is extra-resolvable.*

Corollary 79. (1) *If X is a crowded D^+ -separable space then there is a family \mathcal{G} of dense subspaces of X such that $|\mathcal{G}| = \mathfrak{c}$ and $G \cap G'$ is nowhere dense, for every distinct $G, G' \in \mathcal{G}$.*

(2) *If X is a crowded D^+ -separable space and $\Delta(X) < \mathfrak{c}$ (in particular, if X is countable), then X is extra-resolvable.*

Recently, Garcia-Ferreira and Hrušák [35] constructed (within ZFC) a countable ω -resolvable space which is not extra-resolvable. It follows from Corollary 78 that this provides one more example of a countable space which is not D -separable.

Example 80. *The Garcia-Ferreira-Hrušák example of an ω -resolvable non extra-resolvable space is an example of a countable non- D -separable space.*

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